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CLASSICAL AND QUASI-CLASSICAL ASPECTS OF SUPERSYMMETRIC QUANTUM MECHANICS

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Abstract

A recent development of the studies on classical and quasi-classical properties of supersymmetric quantum mechanics in Witten's version is reviewed. First, classical mechanics of a supersymmetric system is considered. Solutions of the classical equations of motion are given and their properties are discussed in some detail. The corresponding quantum model is constructed by canonical quantization. The quantum model is analyzed by Feynman's path integral within a stationary-phase approximation. A quasi-classical quantization rule is derived, which is applicable when supersymmetry is exact or spontaneously broken.

1. Introduction

The idea of supersymmetry (SUSY) itself is not quite new. It was originally introduced into physics nearly a quarter century ago [1, 2]. It is based on the assumption that there may be environments where distinction between bosons and fermions is irrelevant. A direct and rather successful application of SUSY has been made in the area of nuclear physics [3]. SUSY is essential for the coherent and self-contained formulation of string theory. SUSY is indeed a rich idea that has a wide variety of application in physics [4]. However, as early as 1976, Nicolai [5] suggested that SUSY could be a tool in non-relativistic quantum mechanics. In 1981, Witten [6] utilized SUSY quantum mechanics to simulate spontaneous breaking of SUSY. In the last decade or so, SUSY quantum mechanics has been extensively studied and widely utilized as a tool for studying various problems in quantum and statistical physics.

One of the simple non-trivial models of SUSY quantum mechanics is that of Witten. Besides Witten's original application to SUSY breaking, it has been used in the study of exactly solvable Schrödinger and Dirac problems, for characterizing certain classical stochastic dynamical systems via the Fokker-Planck equation, and even in analyzing properties of semiconductor heterojunctions. For reviews of SUSY methods in quantum and statistical physics, see, e.g., refs. [7, 8].

An interesting property of Witten's model which has attracted considerable attention is the exactness of modified WKB quantization formulas [9, 10, 11] for the so-called shape-invariant potentials [12]. These formulas can be obtained from an evaluation of Feynman's path integral by the stationary-phase-approximation method. In contrast to the usual quadratic expansion of the action about the classical path, the action functional has to be expanded about what we call the quasi-classical path [10, 13]. This brings us to the question as to what the quasi-classical path is and leads to the study of quasi-classical

mechanics before quantization [14]. The aim of this contribution is to review a recent development of the classical and quasi-classical properties of SUSY quantum mechanics in Witten's version.

In Section 2, a classical counterpart of Witten's model is studied. Solutions of the classical equations of motion are discussed in some detail. Section 3 describes the quantum version of the classical model via the canonical quantization. In Section 4, the quasi-classical quantization formulas are discussed. The pseudoclassical analogue of the Bohr-Sommerfeld quantization condition is also presented.

2. Supersymmetric Classical Mechanics

Classical supersymmetric models form a subclass of pseudoclassical mechanics, a notion originally introduced by Casalbuoni [15]. Pseudoclassical mechanics deals with classical systems which are described in terms of Grassmannian variables rather than the usual Cartesian variables. The degrees of freedom which are described by even Grassmann numbers correspond to those usually called bosonic degrees of freedom, whereas those characterized by odd Grassmann numbers give the fermionic degrees of freedom. These classical models may be viewed as the classical limits of quantum models with bosonic and fermionic degrees of freedom [16]. In this Section, we shall discuss supersymmetric classical mechanics, mainly following ref. [14].

Let us start with a pseudoclassical system characterized by the standard supersymmetric Lagrangian [17]

$$L := \frac{1}{2} \dot{x}^2 - \frac{1}{2} \Phi^2(x) + \frac{i}{2} (\bar{\psi} \dot{\psi} - \dot{\bar{\psi}} \psi) - \Phi'(x) \bar{\psi} \psi. \quad (1)$$

Here x describes a bosonic (i.e. an even Grassmann-valued) degree of freedom, and ψ and $\bar{\psi}$ are fermionic (i.e. odd Grassmann-valued) variables. The overbar denotes the Grassmann variant of complex conjugation. The bosonic and fermionic variables are elements of the complex Grassmann algebra CB_2 generated, for example, by the two time-independent odd elements $\psi_0, \bar{\psi}_0$ satisfying the anticommutation relation $\psi_0 \bar{\psi}_0 = -\bar{\psi}_0 \psi_0$:

$$x = a + b \bar{\psi}_0 \psi_0, \quad \psi = c \psi_0 + d \bar{\psi}_0, \quad \bar{\psi} = c^* \bar{\psi}_0 + d^* \psi_0, \quad (2)$$

where $a, b \in \mathbf{R}$ which assures the reality of the bosonic variable $\bar{x} = x$, and $c, d \in \mathbf{C}$. In (1), Φ denotes the so-called SUSY potential which is a differentiable function of x , and $\Phi'(x) = d\Phi(x)/dx$. As x is an even and real Grassmann number as given in (2), so is the potential function. Namely, $\Phi(x) = \Phi(a) + \Phi'(a) b \bar{\psi}_0 \psi_0$.

SUSY of the system characterized by (1) is obvious. The following infinitesimal SUSY transformation,

$$\delta x := \bar{\epsilon} \psi + \bar{\psi} \epsilon, \quad \delta \psi := -(i\dot{x} + \Phi(x))\epsilon, \quad \delta \bar{\psi} = (i\dot{x} - \Phi(x))\bar{\epsilon}, \quad (3)$$

with infinitesimal odd Grassmann numbers, ϵ and $\bar{\epsilon}$, gives rise to a variation of the Lagrangian (1):

$$\delta L = \frac{1}{2} \frac{d}{dt} [(i\dot{x} - i\Phi)\bar{\epsilon}\psi + (i\dot{x} + i\Phi)\bar{\psi}\epsilon]. \quad (4)$$

From this symmetry follow conserved Noether charges:

$$Q := \frac{1}{\sqrt{2}} (\dot{x} - i\Phi(x))\bar{\psi}, \quad \bar{Q} = -\frac{i}{\sqrt{2}} (\dot{x} + i\Phi(x))\psi. \quad (5)$$

The equations of motion which follow from (1) are

$$\ddot{x} = -\Phi(x)\Phi'(x) - \Phi''(x)\bar{\psi}\psi, \quad \dot{\psi} = -i\Phi'(x)\psi, \quad \dot{\bar{\psi}} = i\Phi'(x)\bar{\psi}. \quad (6)$$

From the last two equations, it is evident that $\bar{\psi}(t)\psi(t) = \bar{\psi}(0)\psi(0)$ and $\psi(t) = \psi(0)e^{-2i\varphi(x)}$ with a phase functional $\varphi[x]$. In fact, the phase coincides with what we have called the fermionic phase [13],

$$\varphi[x] := \frac{1}{2} \int_0^t d\tau \Phi'(x(\tau)). \quad (7)$$

As in ref. [13], let us introduce the quasi-classical path, denoted by x_{qc} , as a solution of

$$\ddot{x} = -\Phi(x)\Phi'(x). \quad (8)$$

If we make an ansatz $x(t) = x_{qc}(t) + q(t)\bar{\psi}_0\psi_0$, where x_{qc} and q are real-valued functions of time, and the initial conditions $\psi(0) = \psi_0$, $\bar{\psi}(0) = \bar{\psi}_0$ the solutions of (6) explicitly read [14]

$$\begin{aligned} \psi(t) &= \psi_0 \exp\{-2i\varphi[x_{qc}]\}, & \bar{\psi}(t) &= \bar{\psi}_0 \exp\{2i\varphi[x_{qc}]\}, \\ q(t) &= \frac{\dot{x}_{qc}(t)}{\dot{x}_{qc}(0)} q(0) + \dot{x}_{qc}(t) \int_0^t d\tau \frac{F - \Phi'(x_{qc}(\tau))}{2E - \Phi^2(x_{qc}(\tau))}. \end{aligned} \quad (9)$$

In the above, $E \geq 0$ and $F \in \mathbf{R}$ are constants of integration. They are related to the conserved energy $\mathcal{E} = E + F\bar{\psi}_0\psi_0$ associated with the motion of the bosonic degree of freedom:

$$\begin{aligned} \mathcal{E} &= \frac{1}{2}\dot{x}^2 + \frac{1}{2}\Phi^2(x) + \Phi'(x)\bar{\psi}\psi \\ &= \left[\frac{1}{2}\dot{x}_{qc}^2 + \frac{1}{2}\Phi^2(x_{qc}) \right] + \left[\dot{x}_{qc}q + \Phi(x_{qc})\Phi'(x_{qc})q + \Phi'(x_{qc})\bar{\psi}_0\psi_0 \right]. \end{aligned} \quad (10)$$

Naturally, $E = \frac{1}{2}\dot{x}_{qc}^2 + \frac{1}{2}\Phi^2(x_{qc})$, is the energy conserved along a quasi-classical path.

The quasi-classical equation of motion (8) can be derived from either the Lagrangian of the form,

$$L_{qc} := \frac{1}{2}\dot{x}^2 - \frac{1}{2}\Phi^2(x) = \frac{1}{2}(\dot{x} \pm i\Phi(x))^2 \mp i\Phi(x)\dot{x}, \quad (11)$$

or those of the form,

$$\tilde{L}_{qc}^{\pm} := \frac{1}{2}(\dot{x} \pm i\Phi(x))^2. \quad (12)$$

The second set of Lagrangians are interesting in that they are quadratic in the canonical momentum of x_{qc} ,

$$\xi^{\pm} := \frac{\partial \tilde{L}_{qc}^{\pm}}{\partial \dot{x}} = \dot{x} \pm i\Phi(x) = (\xi^{\mp})^*, \quad (13)$$

which obey along the quasi-classical path the equations $\dot{\xi}^{\pm} = \mp i\Phi'(x_{qc})\xi^{\pm}$ which are identical in form to those for the fermionic variables in (6). Obviously the solutions can be expressed in terms of the fermionic phase: $\xi^{\pm}(t) = \xi^{\pm}(0) \exp\{\mp 2i\varphi[x_{qc}]\}$. However, ξ^{\pm} are variables complex-valued rather than Grassmann-valued. We further note that $E = \frac{1}{2}\xi^{\mp}(t)\xi^{-}(t) = \text{const.}$ Hence, the fermionic solutions can be put into the form

$$\psi(t) = \frac{1}{2E} \xi^+(0)\xi^-(t)\psi_0, \quad \bar{\psi}(t) = \frac{1}{2E} \xi^-(0)\xi^+(t)\bar{\psi}_0. \quad (14)$$

It is also obvious that the conserved Noether charges (5) are given in the form, $Q = \frac{1}{\sqrt{2}}\xi^-(t)\bar{\psi}(t)$, $\bar{Q} = -\frac{1}{\sqrt{2}}\xi^+(t)\psi(t)$. Since the phases of $\xi^-(t)$ and $\bar{\psi}(t)$ are equal and opposite, Q is indeed a constant of motion along the quasi-classical path. The same can be said for \bar{Q} .

Before closing this Section, let us emphasize that the quasi-classical path x_{qc} completely characterizes the solutions (9) once initial conditions are given. Another important point is that along the quasi-classical path x_{qc} the fermionic phase (7) can be explicitly calculated and put into a simple form [10, 13]:

$$\varphi[x_{qc}] = \frac{1}{2} [\text{sgn}(\dot{x}')a(x') - \text{sgn}(\dot{x}'')a(x'')] + n_R a(x_R) - n_L a(x_L). \quad (15)$$

Here we used the notation $x' := x_{qc}(0)$, $\dot{x}' := \dot{x}_{qc}(0)$, $x'' := x_{qc}(t)$, $\dot{x}'' := \dot{x}_{qc}(t)$, $\text{sgn}(x) := x/|x|$, $a(x) := \arcsin(\Phi(x)/\sqrt{2E}) \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, and n_R and n_L denote the numbers of right and left turning points along x_{qc} , respectively. These turning points are given by

$$\Phi^2(x_R) = 2E = \Phi^2(x_L), \quad x_R > x_L. \quad (16)$$

3. Supersymmetric Quantum Mechanics

In order to quantize the pseudoclassical system (1) canonically, we wish now to move to Hamilton's formulation of mechanics. The momenta conjugate to the fermionic variables are

$$\Pi := \frac{\partial L}{\partial \dot{\psi}} = -\frac{i}{2}\bar{\psi}, \quad \bar{\Pi} := \frac{\partial L}{\partial \dot{\bar{\psi}}} = -\frac{i}{2}\psi, \quad (17)$$

which do not depend on ψ and $\bar{\psi}$. Hence, the system is subject to the second-class constraints,¹ $\chi_1 := \Pi + \frac{i}{2}\bar{\psi} \approx 0$ and $\chi_2 := \bar{\Pi} + \frac{i}{2}\psi \approx 0$, which have a non-vanishing Poisson bracket $\{\chi_1, \chi_2\}_P \neq 0$. The pseudoclassical Hamiltonian, which gives rise to equations of motion being equivalent to (6), turns out to be

$$H := \frac{1}{2}(p^2 + \Phi^2(x)) + i\Phi'(x)(\bar{\psi}\bar{\Pi} - \psi\Pi). \quad (18)$$

Under the constraints, Poisson's brackets are not canonical invariants in phase space, so that we have to go over to Dirac's bracket formalism [19]. The Dirac brackets for this constrained system have been calculated [20], the results being

$$\{x, p\}_D = 1, \quad \{\psi, \Pi\}_D = -\frac{1}{2}, \quad \{\bar{\psi}, \bar{\Pi}\}_D = -\frac{1}{2}, \quad \{\psi, \bar{\psi}\}_D = -i, \quad \{\Pi, \bar{\Pi}\}_D = \frac{1}{4} \quad (19)$$

and all others vanish.

Quantization of this system can be achieved by replacing the c -number variables by the corresponding q -number operators and the Dirac brackets by the corresponding (anti-)commutators divided by $(i\hbar)$. Since the fermion variables satisfy the properties, $\psi^2 = \bar{\psi}^2 = \Pi^2 = \bar{\Pi}^2 = 0$, the quantum operators obey the algebra,

$$\begin{aligned} [\hat{x}, \hat{p}]_- &= i\hbar, & [\hat{\psi}, \hat{\Pi}]_+ &= -\frac{i\hbar}{2}, & [\hat{\bar{\psi}}, \hat{\bar{\Pi}}]_+ &= -\frac{i\hbar}{2}, \\ [\hat{\psi}, \hat{\bar{\psi}}]_+ &= \hbar, & \hat{\psi}^2 = 0 &= \hat{\bar{\psi}}^2, & [\hat{\Pi}, \hat{\bar{\Pi}}]_+ &= -\frac{\hbar}{4}, & \hat{\Pi}^2 = 0 &= \hat{\bar{\Pi}}^2. \end{aligned} \quad (20)$$

¹Here the symbol \approx denotes weak equality following Dirac's notation [19].

Besides the well-known Heisenberg algebra for the bosonic operators \hat{x}, \hat{p} , we find, as expected, that $\hat{\psi}, \hat{\bar{\psi}}, \hat{\Pi}$ and $\hat{\bar{\Pi}}$ satisfy the algebra of the fermionic creation and annihilation operators. Obviously, the algebra satisfied by the fermionic operators is isomorphic to that obeyed by Pauli matrices. Hence, as simple representations of the fermionic operators we may employ Pauli matrices: $\hat{\psi} = \sqrt{\hbar}\sigma_-, \hat{\bar{\psi}} = \sqrt{\hbar}\sigma_+, \hat{\Pi} = -(i\sqrt{\hbar}/2)\sigma_+, \hat{\bar{\Pi}} = -(i\sqrt{\hbar}/2)\sigma_-$, and $i(\hat{\bar{\Pi}} - \hat{\psi}\hat{\Pi}) = (\hbar/2)[\sigma_+, \sigma_-] = (\hbar/2)\sigma_3$. Then we arrive at Witten's quantum mechanical Hamiltonian operator acting on $L^2(\mathbf{R}) \otimes \mathbf{C}^2$:

$$\hat{H} := \frac{1}{2}(\hat{p}^2 + \Phi^2(\hat{x}) + \hbar\Phi'(\hat{x})\sigma_3). \quad (21)$$

Similarly, the conserved charges (5) may be converted into quantum operators \hat{Q} and \hat{Q}^\dagger . These charge operators, together with the Hamiltonian operator (21), form the SUSY algebra

$$[\hat{Q}, \hat{Q}^\dagger]_+ = \hat{H}, \quad [\hat{Q}, \hat{H}]_- = 0 = [\hat{Q}^\dagger, \hat{H}]_- \quad (22)$$

On the eigenbasis of σ_3 in \mathbf{C}^2 , the Hamiltonian (21) becomes diagonal, its diagonal elements being the so-called partner Hamiltonians acting on $L^2(\mathbf{R})$:

$$\hat{H}_\pm := \frac{\hat{p}^2}{2} + \frac{\Phi^2(\hat{x})}{2} \pm \frac{\hbar}{2}\Phi'(\hat{x}) \geq 0. \quad (23)$$

SUSY of this system explicates itself in the spectral properties of \hat{H}_\pm :

$$\text{spec}(\hat{H}_+)/\{0\} = \text{spec}(\hat{H}_-)/\{0\}. \quad (24)$$

Namely, \hat{H}_+ and \hat{H}_- have identical spectra except for zero.

As in field theory, SUSY is said to be good if there exists an eigenstate of \hat{H} belonging to a vanishing eigenvalue. Clearly, this state is an eigenstate of either \hat{H}_+ or \hat{H}_- . SUSY is said broken if such a state does not exist. Whether SUSY is good or broken depends on the behavior of the SUSY potential Φ . In order to discriminate between the two cases it is convenient to utilize the Witten index defined by

$$\Delta := \dim \ker(\hat{H}_-) - \dim \ker(\hat{H}_+). \quad (25)$$

Obviously $\Delta = 0$ for broken SUSY, and $\Delta = \pm 1$, if SUSY is good, depending on whether the state belonging to the zero eigenvalue is of \hat{H}_- or \hat{H}_+ . According to a theorem of Atiyah and Singer [21], the Witten index should be independent of the details of the SUSY potential Φ . In fact, it depends only on its asymptotic behavior for $x \rightarrow \pm\infty$:

$$\Delta = \text{sgn}(\Phi(+\infty)) - \text{sgn}(\Phi(-\infty)) = \frac{1}{\pi}(a(x_R) - a(x_L)). \quad (26)$$

For the last equality, we have assumed that there exists a unique set of left and right turning points for a given energy $E \geq 0$. This is always the case if Φ^2 has a single-well structure.

4. Quasi-Classical Approximation

Next, we shall derive quasi-classical formulas for the spectra of the partner Hamiltonians (23). To this end, we shall utilize Feynman's path integral for the kernel (the matrix elements of the time-evolution operator) [10, 13]:

$$\langle x'' | e^{-i(\hbar)^{-1}\hat{H}_\pm t} | x' \rangle = \int \mathcal{D}x \exp \left\{ \frac{i}{\hbar} S_{\text{qc}}[x] \mp i\varphi[x] \right\}, \quad (27)$$

where $S_{\text{qc}}[x] := \frac{1}{2} \int_0^t d\tau (\dot{x}^2 - \Phi^2(x))$. Since it is generally difficult to carry out the path integration of (27), we shall employ the stationary-phase-approximation method to evaluate it. A unique feature of our approach is to expand the action $S^\pm[x] := S_{\text{qc}}[x] \mp \hbar\varphi[x]$ to second order about the quasi-classical path x_{qc} , discussed in Section 2, rather than the classical path. To be more explicit, we expand $S_{\text{qc}}[x]$ about x_{qc} up to second order in $\eta(\tau) := x(\tau) - x_{\text{qc}}(\tau)$, so that we have

$$S^\pm[x] \simeq S_{\text{qc}}[x_{\text{qc}}] \mp \hbar\varphi[x_{\text{qc}}] + \frac{1}{2} \int_0^t d\tau \left[\dot{\eta}^2 - (\Phi^2)''(x_{\text{qc}}(\tau)) \eta^2 \right]. \quad (28)$$

In this approximation, the path integral (27) becomes Gaussian and can be calculated explicitly [10]. Then we calculate again by the stationary-phase approximation the kernel of the resolvent $(E - \hat{H}_\pm)^{-1}$ given as a Laplace transform of the Feynman kernel:

$$\langle x'' | (E - \hat{H}_\pm)^{-1} | x' \rangle = \frac{1}{i\hbar} \int_0^\infty dt \langle x'' | e^{-i(\hbar)^{-1}\hat{H}_\pm t} | x' \rangle e^{iEt/\hbar}, \quad \text{Im } E > 0. \quad (29)$$

As a result, we obtain the supersymmetric version of Gutzwiller's formula [13]

$$\langle x'' | (E - \hat{H}_\pm)^{-1} | x' \rangle \simeq \frac{1}{i\hbar} \frac{\text{fixed } E}{\sqrt{|x''x'|}} \sum_{x_{\text{qc}}} \exp \left\{ \frac{i}{\hbar} W[x_{\text{qc}}] \mp i\varphi[x_{\text{qc}}] - i(n_R + n_L) \frac{\pi}{2} \right\} \quad (30)$$

where $W[x_{\text{qc}}] := \int_{x_{\text{qc}}} dx \sqrt{2E - \Phi^2(x)}$ is Hamilton's characteristic function defined along the quasi-classical path. The phase $(n_R + n_L)\pi/2$, identified with the Maslov phase, arises at the turning points of x_{qc} . In the above the sum has to be made over all quasi-classical paths starting from x' and ending at x'' with a fixed energy E . Using the fermionic phase (15), we may rewrite our result as follows

$$\langle x'' | (E - \hat{H}_\pm)^{-1} | x' \rangle \simeq \frac{\exp \left\{ \mp \frac{1}{2} [\text{sgn}(\dot{x}')a(x') - \text{sgn}(\dot{x}'')a(x'')] \right\}}{i\hbar \sqrt{|x''x'|}} \times \sum_{x_{\text{qc}}} \exp \left\{ \frac{i}{\hbar} W[x_{\text{qc}}] - in_R \left(\frac{\pi}{2} \pm a(x_R) \right) - in_L \left(\frac{\pi}{2} \mp a(x_L) \right) \right\}. \quad (31)$$

From the poles of (31) we can obtain quasi-classical quantization conditions. These poles are easily found by performing the path sum explicitly. Details are given in [10, 13]. Here we only mention that each complete cycle of a periodic path x_{qc} contributes to the Maslov phase an additive term π . Similarly, the contribution of such a cycle to the fermionic phase is given by

$$a(x_R) - a(x_L) = \pi\Delta \quad (32)$$

as n_R and n_L increase by one after each cycle (cf. relations (15) and (26)). The quasi-classical supersymmetric quantization condition thus derived reads for \hat{H}_\pm

$$\oint P_{\text{qc}}(x) dx = 2\pi\hbar \left(n + \frac{1}{2} \pm \frac{\Delta}{2} \right), \quad n \in \mathbf{N}_0, \quad (33)$$

where $p_{qc}(x) := \text{sgn}(x)\sqrt{2E - \Phi^2(x)}$ is the quasi-classical momentum. The integration is to be evaluated for one cycle of the periodic quasi-classical motion. The additional term $\pi\hbar$ on the right-hand side stems from the Maslov phase, just as in the well-known WKB formula, whereas the term $\pi\hbar\Delta$ results from the fermionic phase. The quantization condition (33) coincides with the formula given in ref. [9] if $\Delta = \pm 1$ and the formula proposed in ref. [10, 13] if $\Delta = 0$. A formula basically identical to (33) was earlier derived by Eckhardt [18] from a standard WKB consideration. However, he did not recognize the crucial link between the formula with $\Delta = 0$ and broken SUSY.

The quantization condition (33), resulting in approximate energy eigenvalues E_n^\pm , has some remarkable properties. It leads to the exact ground-state energy, $E_0^- = 0$ for $\Delta = 1$ and $E_0^+ = 0$ for $\Delta = -1$, but $E_0^\pm > 0$ for $\Delta = 0$. It also gives rise to the relation $E_n^+ = E_{n+1}^-$ for $\Delta = 1$, $E_n^- = E_{n+1}^+$ for $\Delta = -1$ and $E_n^\pm = E_n^\pm$ for $\Delta = 0$. In other words, the quasi-classical approximation preserves the exact spectral symmetry (24) between H_+ and H_- . The third and probably most interesting property, however, is that (33) provides exact bound-state spectra for all shape-invariant potentials (i.e. those for which the Schrödinger equation is exactly solvable by the factorization method) [12]. In general, (33) provides better approximation. In particular, for broken SUSY, it has been observed [13] that (33) overestimates the energy eigenvalues, while the usual WKB formula makes underestimation. Thus, (33), together with the standard WKB approximation, can lead to improved energy spectra.

Finally, we wish to present other aspects of the quasi-classical quantization condition (33). Let us consider the classical phase integral for the Grassmann variables [17]

$$\oint (\Pi d\psi + \bar{\Pi} d\bar{\psi}) = \int_0^{T_E} dt (\Pi(t)\dot{\psi}(t) + \bar{\Pi}(t)\dot{\bar{\psi}}(t)), \quad (34)$$

where $T_E := 2 \int_{x_L}^{x_R} dx [2E - \Phi^2(x)]^{-1/2}$ is the period of the bounded quasi-classical motion. Using the relations (17), (6) and (26), we can obtain

$$\begin{aligned} \oint (\Pi d\psi + \bar{\Pi} d\bar{\psi}) &= -\frac{1}{2} \int_0^{T_E} dt \Phi'(x_{qc}) [\bar{\psi}(t), \psi(t)]_- \\ &= -\left(a(x_R) - a(x_L)\right) [\bar{\psi}_0, \psi_0]_- = -\pi\Delta [\bar{\psi}_0, \psi_0]_- \end{aligned} \quad (35)$$

This is a very interesting result. It shows that the Witten index (25) can be directly related to the phase integral (34) which is purely classical. It has been shown by Mañes and Zumino [22] that the Witten index can be derived from a pseudo-classical generalization of Van Vleck's formula. These authors have noted that for an evaluation of the index only classical quantities are need. Nevertheless, their formula (67) in [22] is still a quantum mechanical expression containing \hbar . In contrast, the present formula (35) expresses the Witten index in terms of purely classical quantities.

It is also entertaining to incorporate (35) into (33). Following the canonical quantization procedure shown in Section 3, let us replace the commutator of the classical Grassmann numbers in (35) by $\hbar\sigma_3$. Since σ_3 has eigenvalues ± 1 in the subspaces corresponding to H^\pm , the quasi-classical quantization formula (33) may formally be put into the form

$$\oint (p_{qc} dx + \Pi d\psi + \bar{\Pi} d\bar{\psi}) = 2\pi\hbar \left(n + \frac{1}{2} \right). \quad (36)$$

This formula is the pseudoclassical analogue of the Bohr-Sommerfeld quantization condition.

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